GEOMETRY (II)

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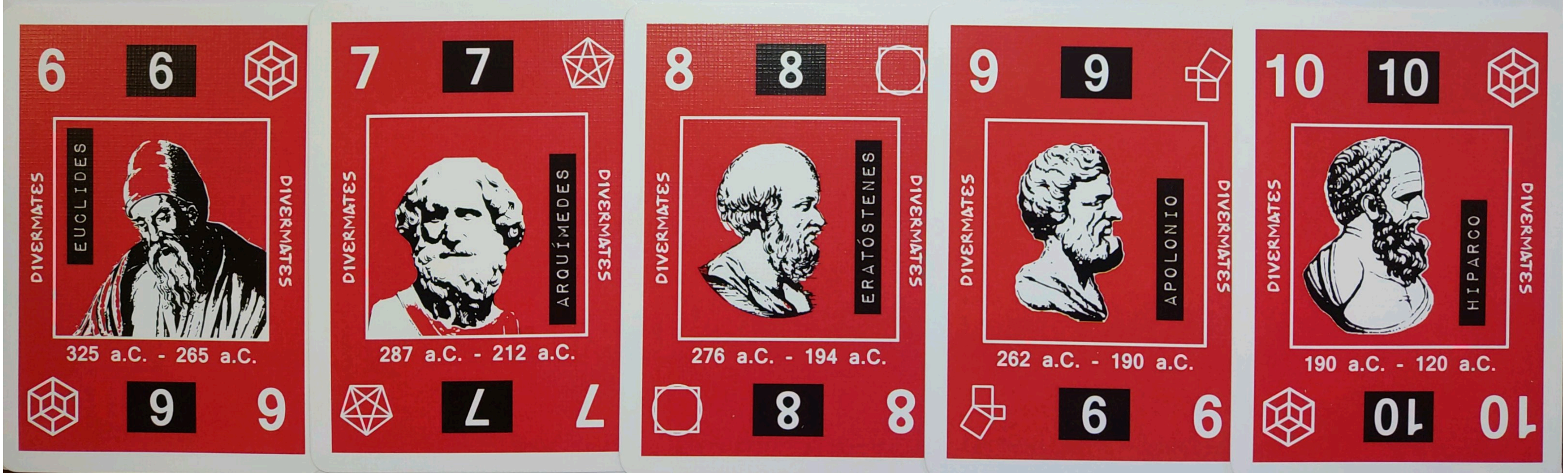
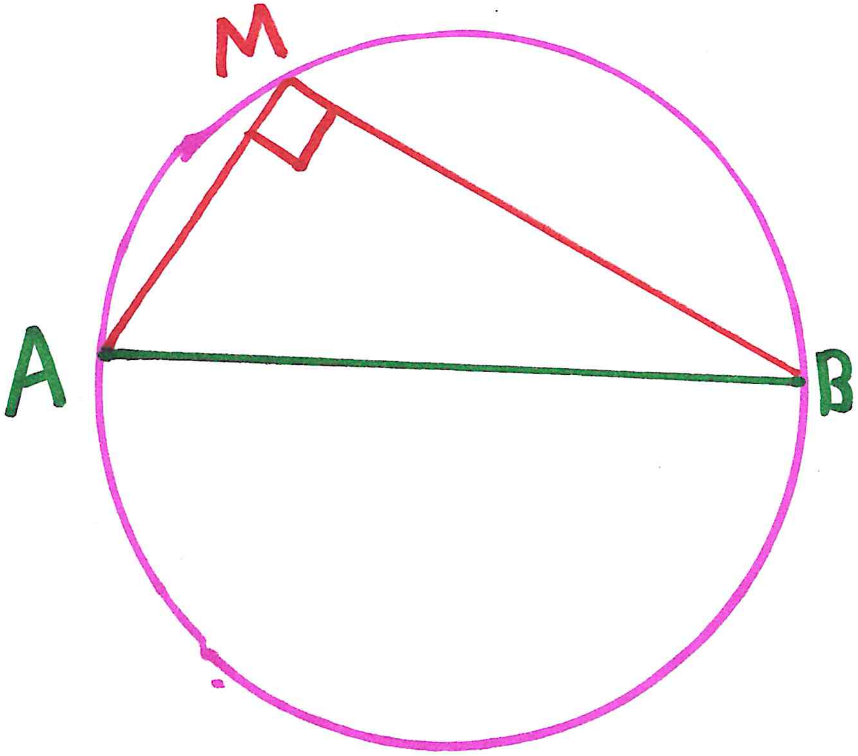


Today : triangles → circles

△ (Geometry I) ○ (Geometry II)

We will focus + extend the following:

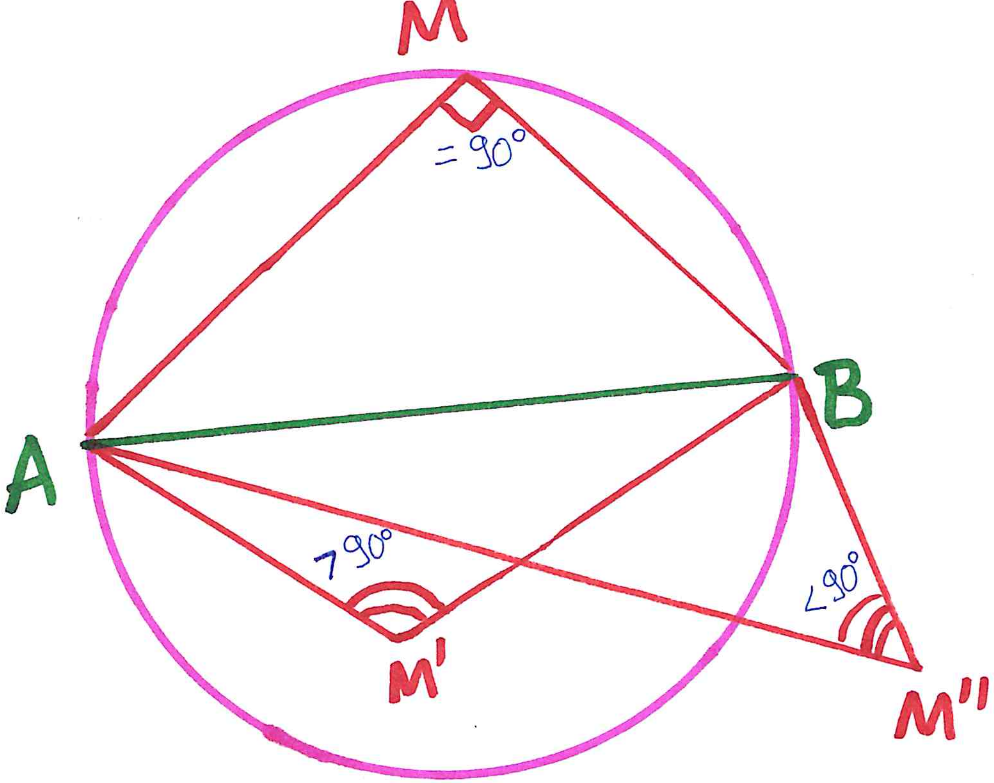


prototype Theorem :

If AB is a diameter of a circle and M is a point of the circle

then $A\hat{M}B = \frac{180°}{2} = 90°$

👁 The converse is also true:



- A point M lies <u>on</u> a circle of diameter AB

⟺ (if and only if)

$\widehat{AMB} = 90°$.

- A point M' lies <u>inside</u> a circle of diameter AB

⟺ $\widehat{AM'B} > 90°$.

- A point M'' lies <u>outside</u> a circle of diameter AB

⟺ $\widehat{AM''B} < 90°$.

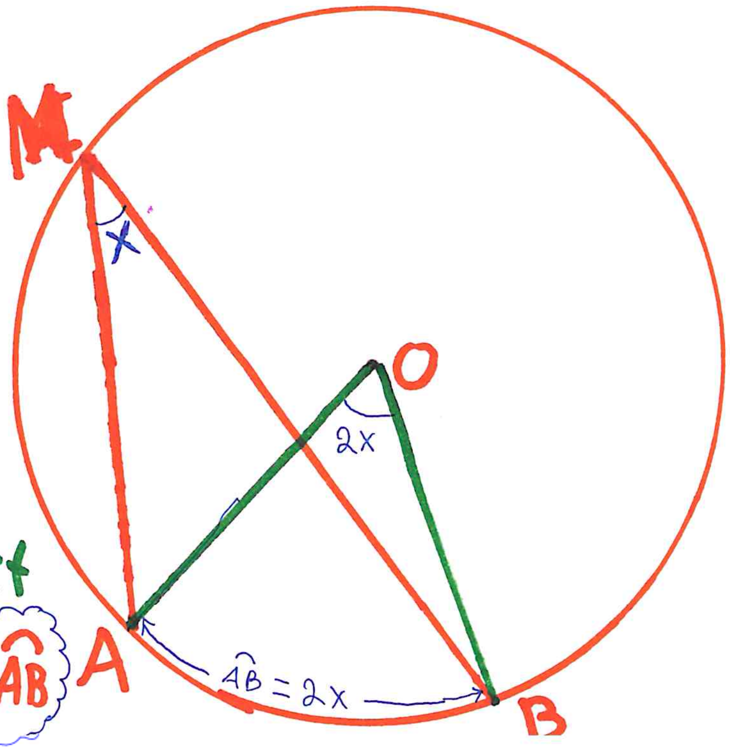
What happens if M lies on a circle of chord AB (not necessarily a diameter)?

- **Def**: (i) A central angle is an angle whose vertex lies at the centre of the circle. Its measure equals the measure of the intercepted arc.

  i.e. $A\hat{O}B = \overset{\frown}{AB}$

  (ii) An inscribed angle is an angle whose vertex lies at a point M on the circle and its legs intersect the circle at some A, B.

  $A\hat{M}B = \frac{1}{2} A\hat{O}B = \frac{1}{2} \overset{\frown}{AB}$

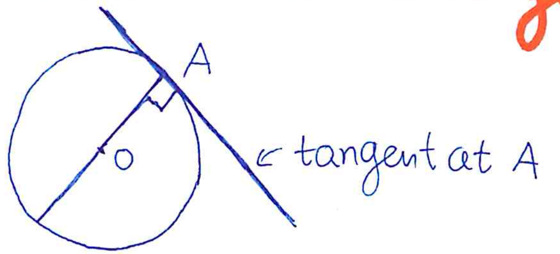


Special Case: If AB is a diameter then:

$$\left\{\begin{array}{l} A\hat{M}B = 90^\circ \\ A\hat{O}B = 180^\circ \\ \overset{\frown}{AB} = 180^\circ \end{array}\right\}$$ so indeed we have

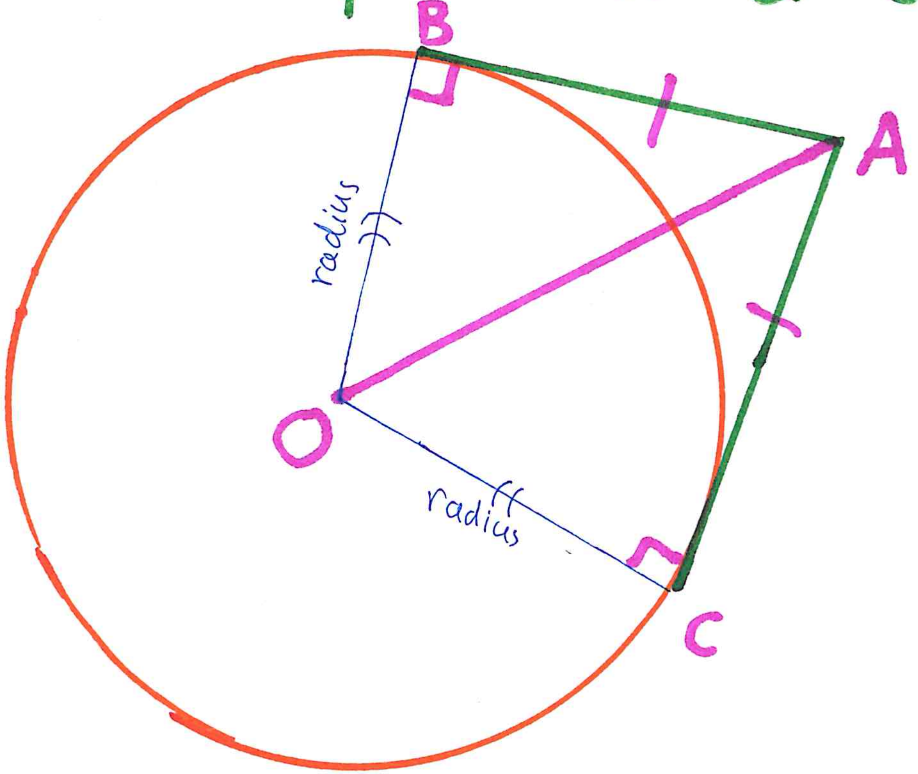
$$A\hat{M}B = \frac{1}{2} A\hat{O}B = \frac{1}{2} \overset{\frown}{AB}.$$

- **Def:** A line that has exactly 1 common point with a circle is called a tangent line to the circle.



Fact 1: A tangent at a point A on a circle is perpendicular to the diameter passing through A

Fact 2: Through a point A outside of a circle, exactly 2 tangent lines can be drawn. The corresponding tangent segments drawn from an exterior point to a circle are equal.



i.e. $AB = AC$.

proof: $O\hat{B}A = O\hat{C}A$

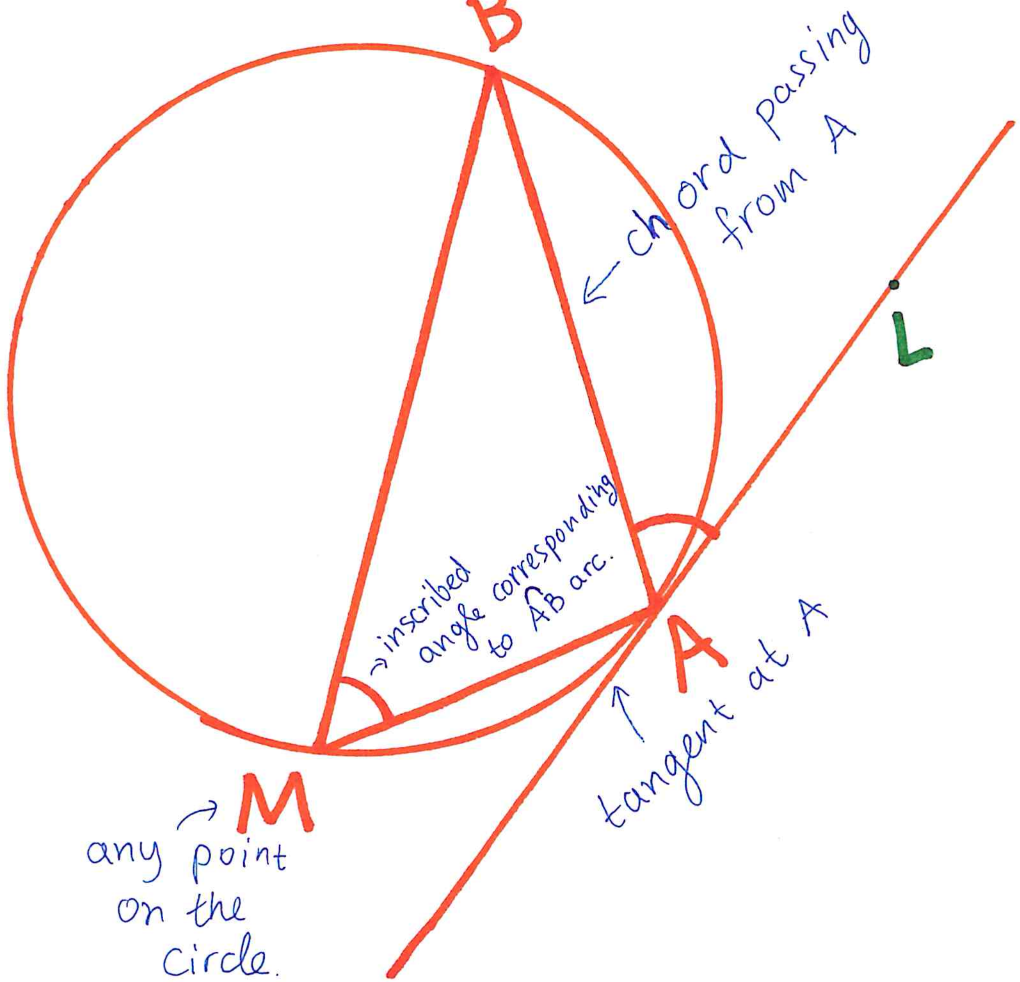
because:
- $OB = OC$ (radii)
- OA common
- $O\hat{B}A = O\hat{C}A = 90°$

⇓ pythagoras' Thm

- $AB = AC$

(from this we also get that $O\hat{A}C = O\hat{A}B$, $A\hat{O}C = A\hat{O}B$)

Fact 3:

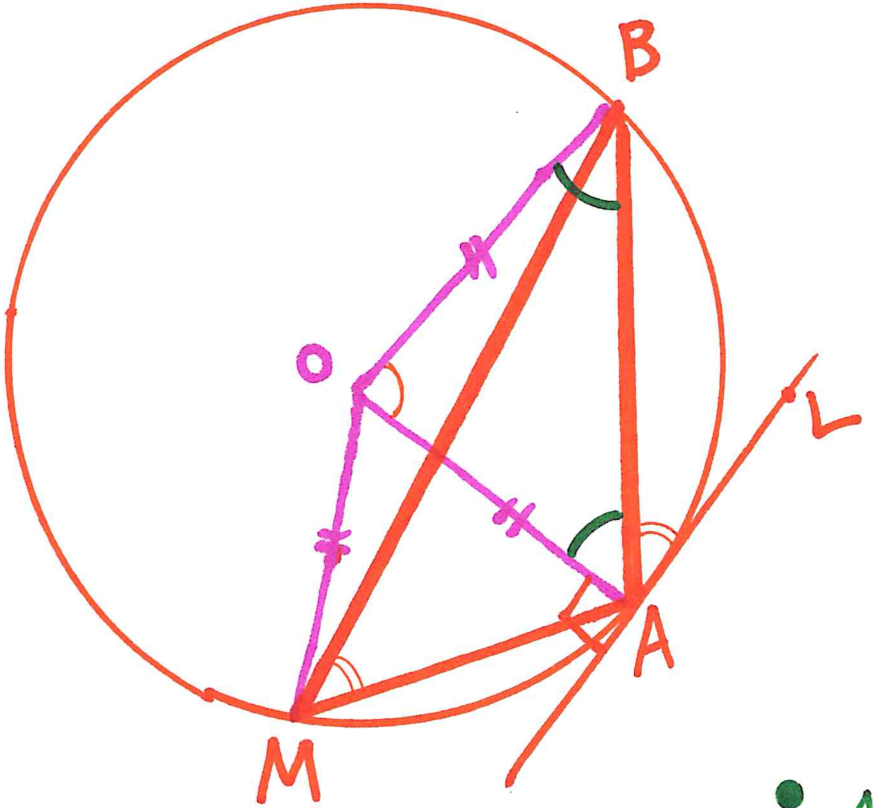


If AB is a chord of a circle and AL is a tangent to the circle at the point A, then **for every point M on the circle** we have

$$\hat{BAL} = \hat{AMB}.$$

↑ angle between the tangent AL and the chord AB

↑ inscribed angle with vertex at M, corresponding to $\hat{AB}$.



proof:

- $\left.\begin{array}{l}\widehat{AOB} \text{ central angle} \\ \widehat{AMB} \text{ inscribed angle}\end{array}\right\}$ corresponding to the arc $\overset{\frown}{AB}$:

$$\Rightarrow \widehat{AMB} = \frac{1}{2}\widehat{AOB} \quad (1)$$

- $\overset{\triangle}{AOB}$ isosceles (since OA = OB as radii)

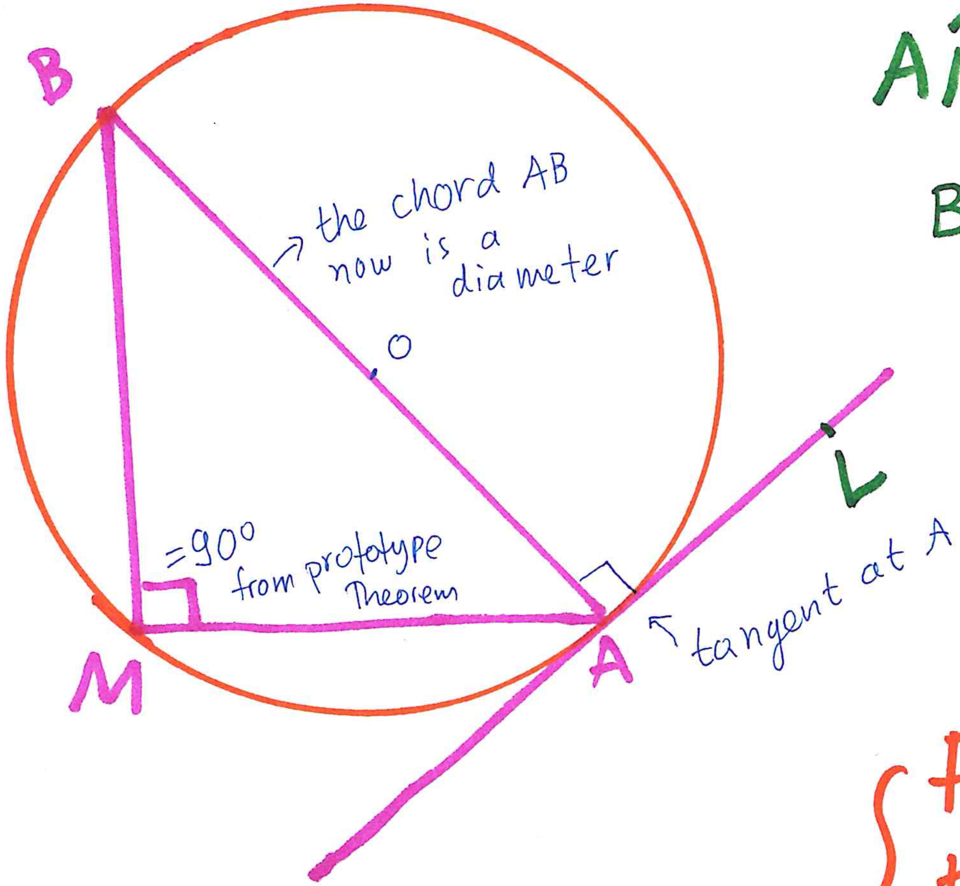
$$\underset{OA=OB}{\Longrightarrow} \widehat{OBA} = \widehat{BAO} \Rightarrow \widehat{AOB} = 180° - \widehat{OBA} - \widehat{BAO} =$$
$$= 180° - 2\widehat{BAO} \quad (2)$$

- AL tangent at A $\Rightarrow \widehat{BAL} = 90° - \widehat{BAO}$ (3)

$$\blacktriangleright \underset{(1),(2)}{} \widehat{AMB} = \frac{1}{2}(180° - 2\widehat{BAO}) = 90° - \widehat{BAO} \underset{(3)}{=} \widehat{BAL}$$

Special Case:

If AB is a diameter then (from Prototype Theorem)



$A\hat{M}B = 90°$. (since AB is a diameter)

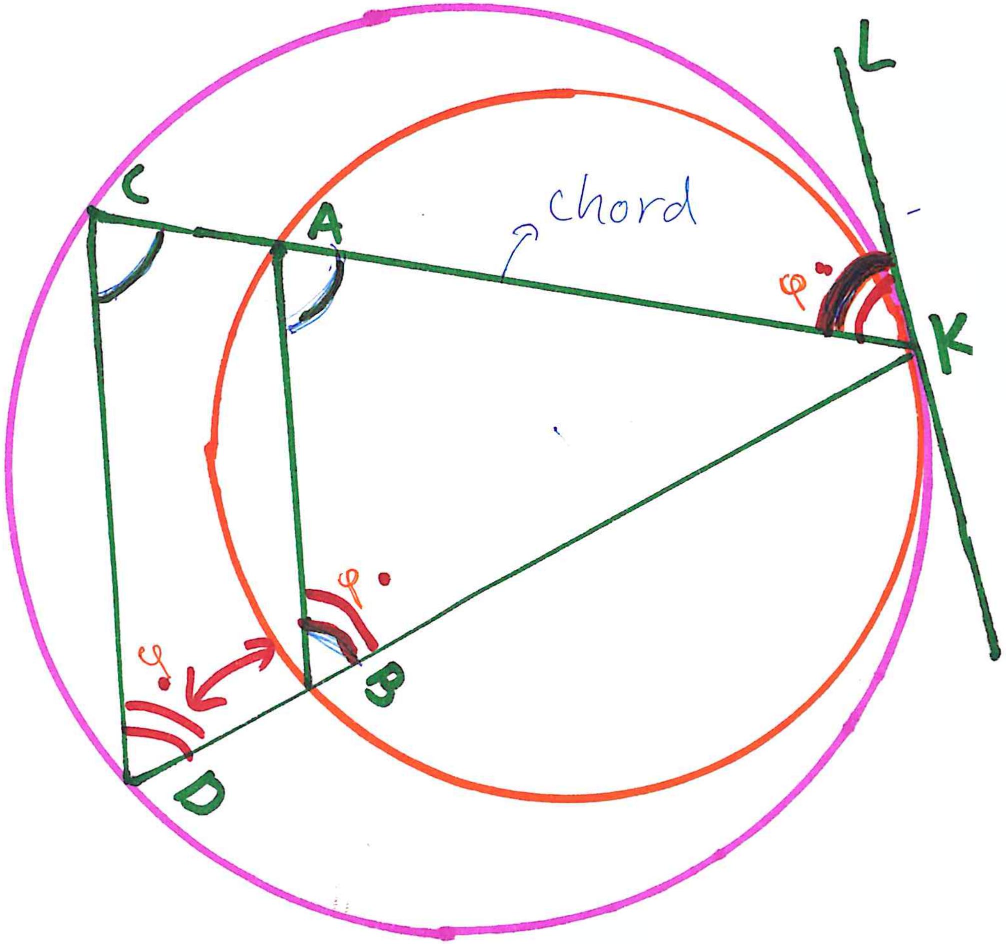
$B\hat{A}L = 90°$ (since AB is a diameter + AL is a tangent at A)

$\Downarrow$

$$A\hat{M}B = B\hat{A}L.$$

Fact 3 (which is a useful theorem) generalizes this special case from diameters AB to **all** chords AB!

Exercise 1 : Two circles are internally tangent at K. Two lines passing through K intersect the two circles at A, C and B, D respectively. Prove that $AB \parallel CD$.



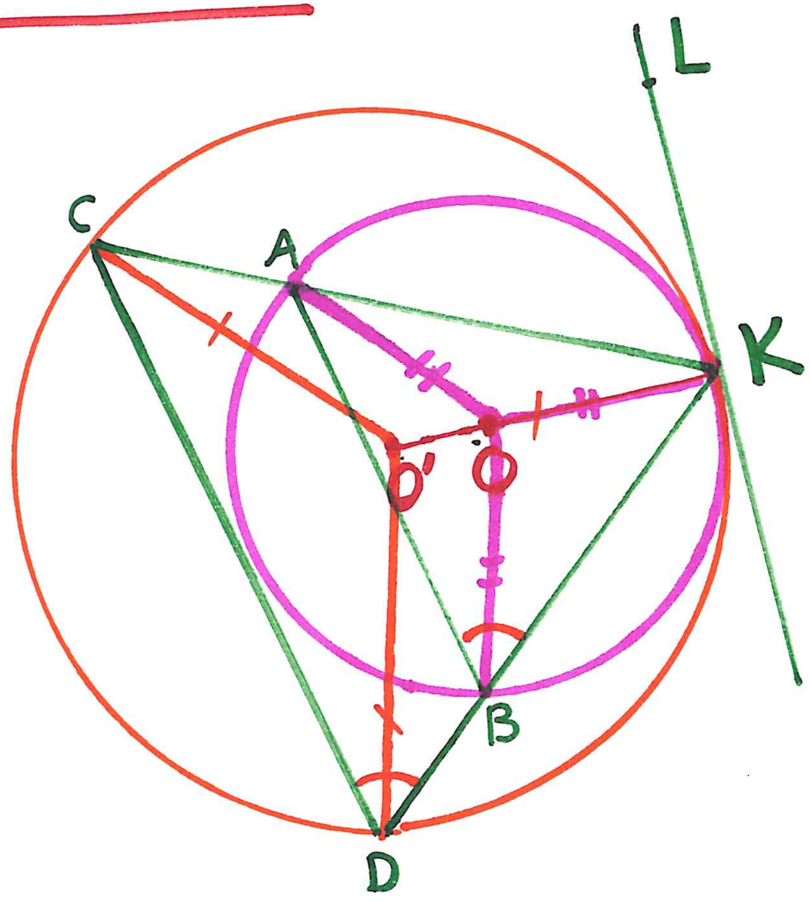
Solution #1: Let KL be the common tangent line to the 2 circles. Then (using Fact 3):

- $L\hat{K}A = K\hat{B}A$ (inscribed angle corresponding to AK)
- $L\hat{K}A = L\hat{K}C = K\hat{D}C$ (inscribed angle corresponding to CK)

Conclusion: $\boxed{K\hat{B}A = K\hat{D}C} \Rightarrow AB \parallel CD$.

Comment: In class it was asked if $LK \parallel AB$.
This is not always true, because $L\hat{K}C \neq K\hat{A}B$ in general.
(there was confusion because there was a typo in the letters that were used in the proof).

Solution #2:



Let O, O' be the centres of the 2 circles.

- First notice that O', O, K are collinear because $\begin{cases} O'K \perp KL \\ OK \perp KL \end{cases}$ and since KL is tangent at K.

- Goal: Show that

$$\boxed{C\hat{D}K = A\hat{B}K}$$

It suffices to show that:

$$\begin{cases} C\hat{D}O' = A\hat{B}O & (1) \\ \text{and} & \\ O'\hat{D}K = O\hat{B}K & (2) \end{cases}$$

proof of (1): • $C\hat{O'}D$ is a central angle corresponding to $\overset{\frown}{CD}$

$\Rightarrow C\hat{O'}D = \overset{\frown}{CD} = 2\,C\hat{K}D$ (inscribed angle corr. to $\overset{\frown}{CD}$)

- $A\hat{O}B$ is a central angle corresponding to $\overset{\frown}{AB}$

$\Rightarrow A\hat{O}B = \overset{\frown}{AB} = 2A\hat{K}B$ (inscribed angle corr. to $\overset{\frown}{AB}$)

$\xRightarrow{A\hat{K}B = C\hat{K}D}$ $\boxed{A\hat{O}B = C\hat{O'}D}$ (*)

- Since $\overset{\triangle}{CO'D}$ is isosceles with $CO' = O'D$ (radii), we have $C\hat{D}O' = D\hat{C}O'$ and so:

$2C\hat{D}O' + C\hat{O'}D = 180 \Rightarrow \boxed{C\hat{O'}D = 180 - 2C\hat{D}O'}$ (**)

- Similarly, since $\overset{\triangle}{AOB}$ is isosceles with $AO = OB$ (radii) we get $A\hat{B}O = B\hat{A}O$ and so:

$2A\hat{B}O + A\hat{O}B = 180 \Rightarrow \boxed{A\hat{O}B = 180 - 2A\hat{B}O}$ (***)

▶ (*), (**), (***): $\cancel{180} - 2A\hat{B}O = \cancel{180} - 2C\hat{D}O' \Rightarrow A\hat{B}O = C\hat{D}O \Rightarrow$ (1) ✓

proof of ②: • $D\hat{O'}K$ is isosceles with $DO' = O'K$ (radii)

$\Rightarrow$ $O'\hat{D}K = O'\hat{K}D$ ⊛'

• $B\hat{O}K$ is isosceles with $BO = OK$ (radii)

$\Rightarrow$ $O\hat{B}K = O\hat{K}B$ ⊛''

▸ ⊛', ⊛'' + "$O'\hat{K}D = O\hat{K}B$" (because O, O', K : collinear)

⇓

$O'\hat{D}K = O\hat{B}K$ $\Rightarrow$ ② ✓

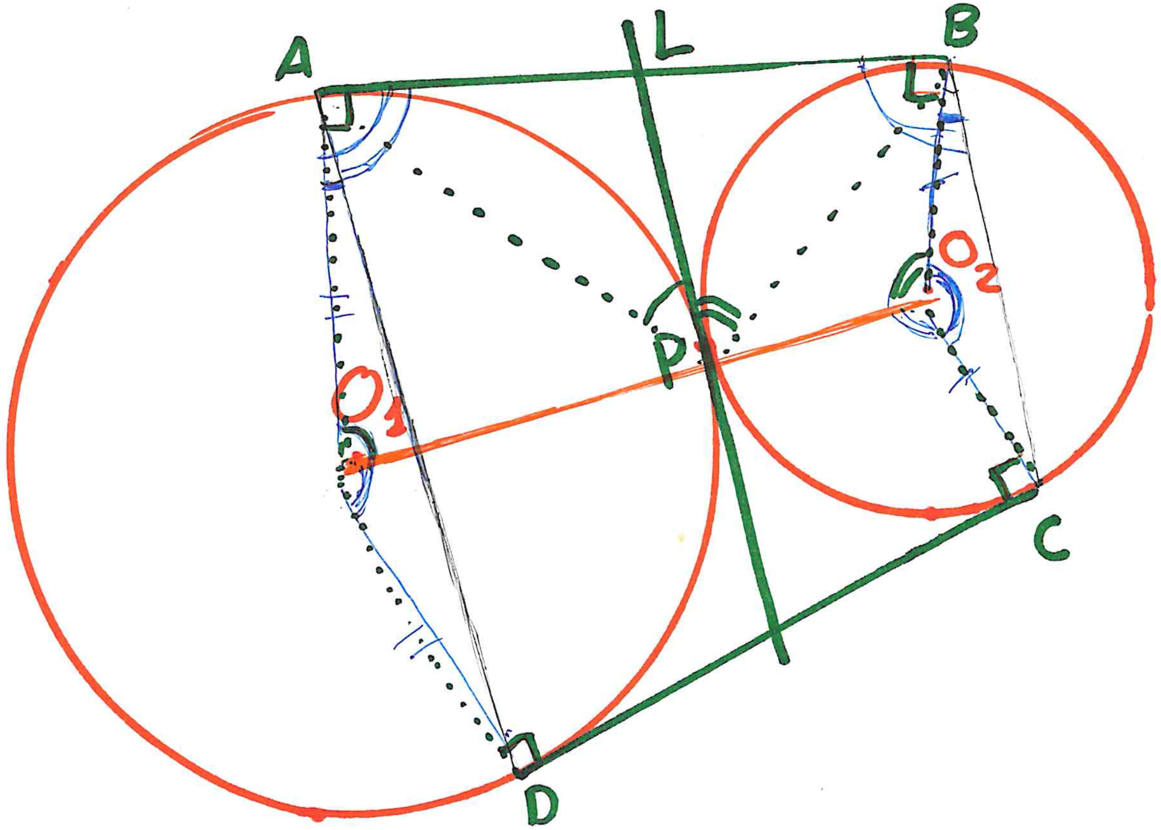
▶ By ① + ②: $C\hat{D}K = C\hat{D}O' + O'\hat{D}K = A\hat{B}O + O\hat{B}K = A\hat{B}K$ which proves the goal, and implies that $AB \parallel CD$.

(This solution was suggested by a student in class. It is longer, but it doesn't use Fact 3.)

Exercise2 : Circles C_1 and C_2 having centres at O_1 and O_2 are externally tangent at P. Denote by AB and CD the two common tangents to these circles so that A, D lie on C_1 and B, C lie on C_2. Show that:

(a) AD // BC (seen in class)

(b) AP ⊥ BP. (Homework)



Solution: (a) O_1, O_2, P : collinear.

- $\overset{\triangle}{O_1AD}$
- $\overset{\triangle}{O_2BC}$

} isosceles triangles (sides: radii) ①

Claim: $A\hat{O_1}D = B\hat{O_2}C$ ②

Indeed, $AO_1 \parallel BO_2$ since $AB \perp O_1A$, $AB \perp O_2B$. $\Rightarrow A\hat{O_1}P = 180 - B\hat{O_2}P$

Similarly $DO_1 \parallel CO_2$ since $DC \perp O_1D$, $DC \perp O_2C$ $\Rightarrow P\hat{O_1}D = 180 - P\hat{O_2}C$

$$\Rightarrow \begin{cases} A\hat{O_1}D = A\hat{O_1}P + P\hat{O_1}D \\ B\hat{O_2}C = 360^\circ - B\hat{O_2}P - P\hat{O_2}C = 360^\circ - (180 - A\hat{O_1}P) - (180 - P\hat{O_1}D) \end{cases}$$

$$= A\hat{O_1}P + P\hat{O_1}D = A\hat{O_1}D.$$

From ① + ②: $O_1\hat{A}D = O_2\hat{B}C \Rightarrow O_1\hat{A}B + A\hat{B}C = 180^\circ \Rightarrow AD \parallel BC$.

(b) Let L be the intersection of the tangent with AB. Then:

$$\left.\begin{cases} A\hat{O_1}P = 2\,A\hat{P}L \\ B\hat{P}O_2 = 2\,L\hat{P}B \end{cases}\right\} \quad (3)$$

Since $AO_1 \parallel BO_2$: $A\hat{O_1}P + B\hat{P}O_2 = 180°$

$$\overset{(3)}{\Longrightarrow} 2\,A\hat{P}L + 2\,L\hat{P}B = 180° \Rightarrow A\hat{P}L + L\hat{P}B = 90°$$

$$\Rightarrow A\hat{P}B = 90°$$

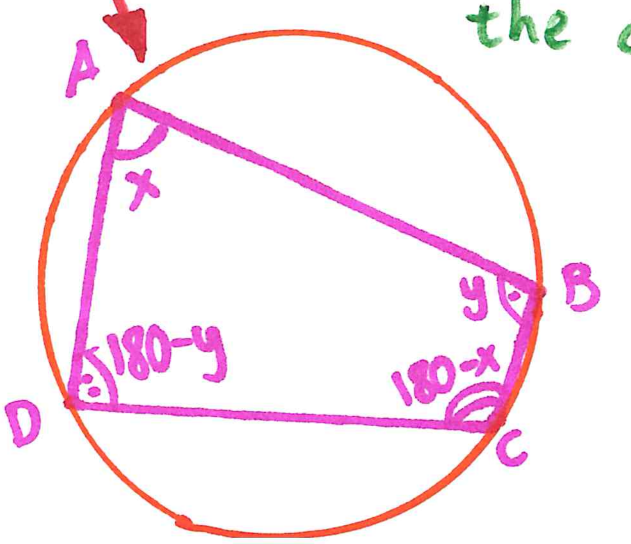
$$\Rightarrow AP \perp BP.$$

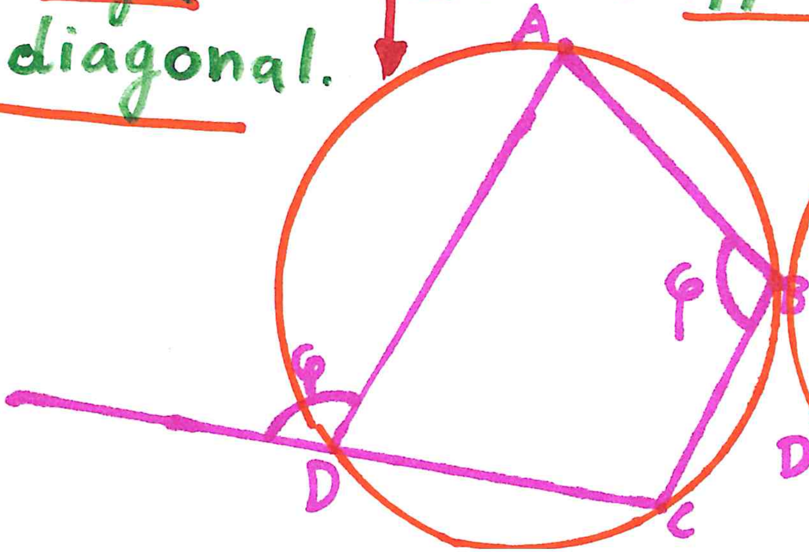
- **Def:** A convex quadrilateral is called cyclic $\Leftrightarrow$ its 4 vertices lie on a circle

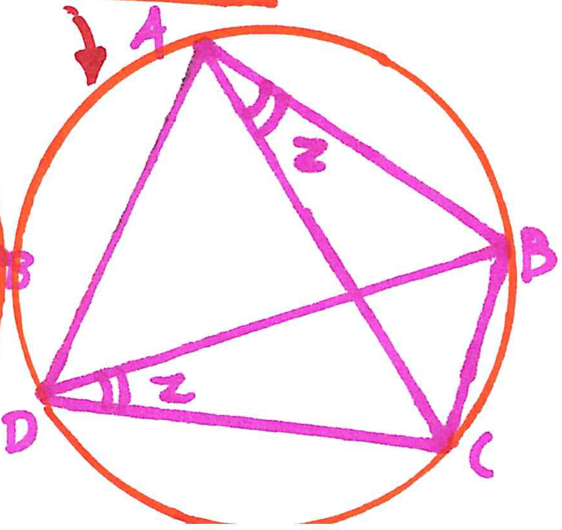
Thm: $\Leftrightarrow$ (1) The sum of 2 opposite sides is $180°$

$\Leftrightarrow$ (2) One angle formed by 2 consecutive sides equals the external angle formed by the other 2.

$\Leftrightarrow$ (3) The angle between one side and a diagonal equals the angle between the opposite side and the other diagonal.

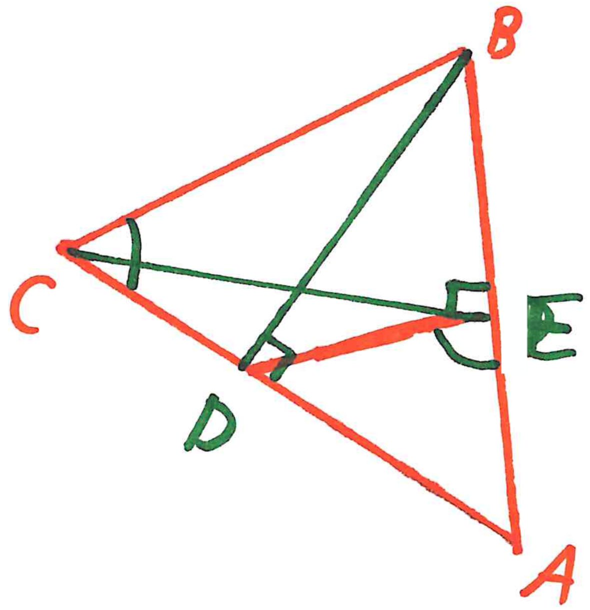






Exercise 3: Let BD and CE be altitudes of a triangle $\overset{\triangle}{ABC}$. Prove that if $DE \parallel BC$ then $AB = AC$.

↑
(Homework)



Solution: BCDE is cyclic since $\widehat{BEC} = \widehat{CDB} = 90°$.

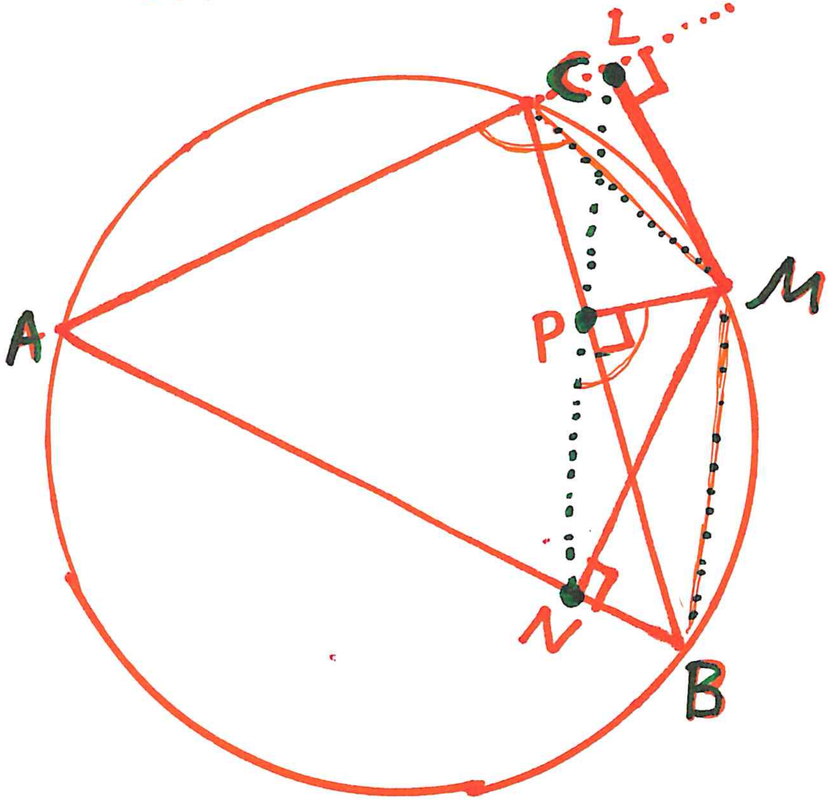
$\Rightarrow \widehat{AED} = \widehat{ACB}$

Since $DE \parallel BC$: $\widehat{AED} = \widehat{ABC}$

$\Big\} \Rightarrow \widehat{ACB} = \widehat{ABC} \Rightarrow \overset{\triangle}{ABC}$ isosceles with $AB = AC$.

Exercise 4 (Simpson's line).

Let M be a point on the circumcircle of a triangle $\overset{\Delta}{ABC}$. Prove that the feet of perpendiculars from M to the sides AB, BC and CA are collinear.



Solution: **Goal:** Show that $\widehat{NPM} + \widehat{MPL} = 180°$.

1: MCAB is a cyclic quadrilateral, so

$$\underbrace{\widehat{MBA}} + \widehat{ACM} = \underbrace{\underbrace{\widehat{MBN}} + \widehat{ACM}} = 180° \qquad (*)$$

2: MPNB is a cyclic quadrilateral, because $\widehat{MPB} = \widehat{MNB} = 90°$. Hence: $\widehat{MBN} + \widehat{NPM} = 180°$

$$\Rightarrow \widehat{NPM} \overset{(*)}{=} \underbrace{180° - \widehat{MBN}} = \widehat{ACM} \qquad (**)$$

3: MLCP is cyclic because $\widehat{CLM} + \widehat{CPM} = 90° + 90° = 180°$.

$\Rightarrow$

$$\widehat{MPL} = \widehat{MCL} = 180^\circ - \widehat{ACM} \quad (***)$$

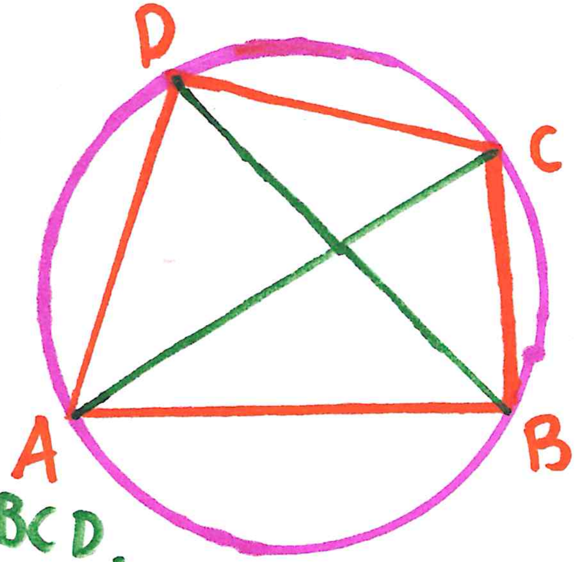
$$\xrightarrow[(***)]{(**)} \widehat{NPM} + \widehat{MPL} = \widehat{ACM} + 180^\circ - \widehat{ACM} = 180^\circ$$

$\Rightarrow L, P, N$: collinear!

Exercise 5 (Ptolemy's Theorem) ← Homework

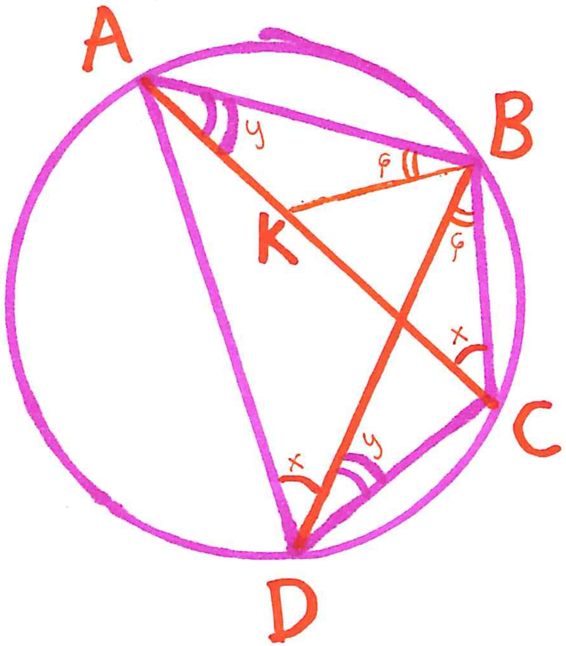
Prove that for each cyclic quadrilateral the product of the lengths of its diagonals is equal to the sum of the products of the lengths of the pairs of opposite sides, i.e. if its vertices are A, B, C, D in order then:

$$AC \cdot BD = AB \cdot CD + BC \cdot AD \quad (*)$$



Remark: The converse is true: If (*) holds for a quadrilateral ABCD, then it is cyclic.

Solution: • Without loss of generality we may assume that $\hat{CBD} \leq \hat{DBA}$.



Then, we can choose a point K on AC such that:

$$\boxed{\hat{CBD} = \hat{ABK}} \quad (1)$$

$$\Rightarrow \begin{cases} \hat{ABK} + \hat{CBK} = \hat{ABC} \\ \hat{ABD} + \hat{CBD} = \hat{ABC} \end{cases}$$

$\Downarrow$ (1) + subtract the 2 equalities

$$\boxed{\hat{CBK} = \hat{ABD}} \quad (2)$$

• Since ABCD cyclic $\Rightarrow$ $\boxed{\hat{CAB} = \hat{CDB}}$ (3), $\boxed{\hat{ADB} = \hat{ACB}}$ (4)

(1), (3) $\Longrightarrow$ $\hat{DBC} \approx \hat{ABK}$ (similar triangles)

$$\Longrightarrow \frac{CD}{BD} = \frac{AK}{AB} \iff \boxed{AK \cdot BD = AB \cdot CD} \quad (5)$$

(2), (4) $\Longrightarrow$ $\hat{KBC} \approx \hat{ABD}$ (similar triangles)

$$\Longrightarrow \frac{CK}{BC} = \frac{DA}{BD} \iff \boxed{CK \cdot BD = BC \cdot DA} \quad (6)$$

$$\xrightarrow{(5)+(6)} AK \cdot BD + CK \cdot BD = AB \cdot CD + BC \cdot DA$$

$$\iff (AK + CK) \cdot BD = AB \cdot CD + BC \cdot DA$$

$$\iff \boxed{AC \cdot BD = AB \cdot CD + BC \cdot DA}$$